



# Typical Rényi dimensions of measures. The cases: $q = 1$ and $q = \infty$

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## Abstract

We study the typical behaviour (in the sense of Baire's category) of the  $q$ -Rényi dimensions  $\underline{D}_\mu(q)$  and  $\overline{D}_\mu(q)$  of a probability measure  $\mu$  on  $\mathbb{R}^d$  for  $q \in [-\infty, \infty]$ . Previously we found the  $q$ -Rényi dimensions  $\underline{D}_\mu(q)$  and  $\overline{D}_\mu(q)$  of a typical measure for  $q \in (0, \infty)$ . In this paper we determine the  $q$ -Rényi dimensions  $\underline{D}_\mu(q)$  and  $\overline{D}_\mu(q)$  of a typical measure for  $q = 1$  and for  $q = \infty$ . In particular, we prove that a typical measure  $\mu$  is as irregular as possible: for  $q = \infty$ , the lower Rényi dimension  $\underline{D}_\mu(q)$  attains the smallest possible value, and for  $q = 1$  and  $q = \infty$  the upper Rényi dimension  $\overline{D}_\mu(q)$  attains the largest possible value.

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## 1. Statement of results

For a Borel probability measure  $\mu$  on  $\mathbb{R}^d$  and  $q \in [-\infty, \infty]$ , we define the lower and upper  $q$ -Rényi dimensions of  $\mu$  by

$$\underline{D}_\mu(q) = \liminf_{r \searrow 0} \frac{1}{q-1} \frac{\log \int_{\text{supp } \mu} \mu(B(x, r))^{q-1} d\mu(x)}{\log r} \quad \text{for } q \in \mathbb{R} \setminus \{1\},$$

$$\overline{D}_\mu(q) = \limsup_{r \searrow 0} \frac{1}{q-1} \frac{\log \int_{\text{supp } \mu} \mu(B(x, r))^{q-1} d\mu(x)}{\log r} \quad \text{for } q \in \mathbb{R} \setminus \{1\},$$

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$$\underline{D}_\mu(1) = \liminf_{r \searrow 0} \frac{\int_{\text{supp } \mu} \log \mu(B(x, r)) d\mu(x)}{\log r},$$

$$\overline{D}_\mu(1) = \limsup_{r \searrow 0} \frac{\int_{\text{supp } \mu} \log \mu(B(x, r)) d\mu(x)}{\log r},$$

and

$$\underline{D}_\mu(-\infty) = \liminf_{r \searrow 0} \frac{\log \inf_{x \in \text{supp } \mu} \mu(B(x, r))}{\log r},$$

$$\overline{D}_\mu(-\infty) = \limsup_{r \searrow 0} \frac{\log \inf_{x \in \text{supp } \mu} \mu(B(x, r))}{\log r},$$

$$\underline{D}_\mu(\infty) = \liminf_{r \searrow 0} \frac{\log \sup_{x \in \text{supp } \mu} \mu(B(x, r))}{\log r},$$

$$\overline{D}_\mu(\infty) = \limsup_{r \searrow 0} \frac{\log \sup_{x \in \text{supp } \mu} \mu(B(x, r))}{\log r}.$$

The Rényi dimensions were essentially introduced by Rényi [9,10] in 1960 as a tool for analyzing various problems in information theory. Indeed, for a probability vector  $\mathbf{p} = (p_1, \dots, p_n)$  and  $q \in \mathbb{R}$ , Rényi defined the  $q$ -entropy  $H_{\mathbf{p}}(q)$  of  $\mathbf{p}$  by  $H_{\mathbf{p}}(q) = \frac{1}{1-q} \log \sum_i p_i^q$  for  $q \neq 1$  and  $H_{\mathbf{p}}(1) = -\sum_i p_i \log p_i$ . Observe that l'Hospital's rule shows that  $H_{\mathbf{p}}(q) \rightarrow H_{\mathbf{p}}(1)$  as  $q \rightarrow 1$ , and the  $q$ -entropies  $H_{\mathbf{p}}(q)$  can therefore be regarded as natural generalizations of the usual entropy  $H_{\mathbf{p}}(1) = -\sum_i p_i \log p_i$  of  $\mathbf{p}$ . The entropies  $H_{\mathbf{p}}(q)$  are discussed in detail by Rényi in [11, Chapter 9].

The main significance of the Rényi dimensions, is their relationship with the multifractal spectrum of  $\mu$ . For a probability measure  $\mu$  on  $\mathbb{R}^d$ , the local dimension  $\mu$  at the point  $x$  is defined by

$$\dim_{\text{loc}}(x; \mu) = \lim_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r}. \quad (1.1)$$

We define the Hausdorff multifractal spectrum function,  $f_\mu$ , of  $\mu$  as the Hausdorff dimension of the level sets of the local dimension of  $\mu$ , i.e. we put

$$f_\mu(\alpha) = \dim \left\{ x \in \mathbb{R}^d \mid \lim_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\}, \quad \alpha \geq 0, \quad (1.2)$$

where  $\dim$  denotes the Hausdorff dimension. Next, recall that the Legendre transform  $\varphi^*$  of a function  $\varphi: \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $\varphi^*(x) = \inf_y (xy + \varphi(y))$ . In the 1980s it was conjectured in the physics literature [4,5] that for “good” measures the following result, relating the multifractal spectrum function  $f_\mu$  to the Legendre transform of the Rényi dimensions, holds: namely (1) that the upper and lower Rényi dimensions coincide, i.e.

$$\underline{D}_\mu(q) = \overline{D}_\mu(q), \quad (1.3)$$

for all  $q \in \mathbb{R}$ , and (2) that the multifractal spectrum function  $f_\mu$  coincides with the Legendre transform of the function  $\tau_\mu: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $\tau_\mu(q) = (1-q)\underline{D}_\mu(q) = (1-q)\overline{D}_\mu(q)$ , i.e.

$$\dim \left\{ x \in \mathbb{R}^d \mid \lim_{r \searrow 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha \right\} = \tau_\mu^*(\alpha), \quad (1.4)$$

for all  $\alpha \geq 0$ . This result is known as the Multifractal Formalism. During the 1990's there has been an enormous interest in verifying the Multifractal Formalism and computing the multifractal spectra of measures in the mathematical literature, and within the last 8 or 9 years the multifractal spectra of various classes of measures in Euclidean space  $\mathbb{R}^d$  exhibiting some degree of self-similarity have been computed rigorously, cf. [1] and the references therein. In particular, it has been proved that many “nice” measures, including (deterministic and stochastic) self-similar measures and certain classes of invariant measures of dynamical systems, satisfy the Multifractal Formalism, cf. [1,8].

In this paper we study the Rényi dimensions of a typical measure in the sense of Baire. One of the consequences of our main results is that a typical measure fails to satisfy part (1.3) of the Multifractal Formalism in a very spectacular way. For a compact subset  $K$  of  $\mathbb{R}^d$ , we denote the family of Borel probability measures on  $K$  by  $\mathcal{P}(K)$  and we equip  $\mathcal{P}(K)$  with the weak topology. We will say that a typical probability measure on  $K$  has property  $\mathcal{P}$ , if the set of probability measures that do not have property  $\mathcal{P}$ , i.e. if the set

$$\{\mu \in \mathcal{P}(K) \mid \mu \text{ does not have property } \mathcal{P}\},$$

is of the first category with respect to the weak topology on  $\mathcal{P}(K)$ .

In [6] we found the  $q$ -Rényi dimensions of a typical measure for  $q \in (0, \infty)$ , and the purpose of this paper is to complement this result by determining the  $q$ -Rényi dimensions of a typical measure for  $q = 1$  and  $q = \infty$ . However, before we state this result it is instructive to recall the result from [6] giving the  $q$ -Rényi dimensions of a typical measure for  $q \in (0, \infty)$ . To state this result we begin with a few definitions. For a subset  $E$  of  $\mathbb{R}^d$ , we denote the lower box dimension of  $E$  and the upper box dimension of  $E$  by  $\underline{\dim}_{\mathcal{B}}(E)$  and  $\overline{\dim}_{\mathcal{B}}(E)$ , respectively; the reader is referred to [1] for the definitions of the box dimensions. Also, for a subset  $K$  of  $\mathbb{R}^d$  and  $x \in K$  we define the lower local box dimension of  $K$  at  $x$  and the upper local box dimension of  $K$  at  $x$  by

$$\underline{\dim}_{\mathcal{B},\text{loc}}(x, K) = \lim_{r \searrow 0} \underline{\dim}_{\mathcal{B}}(K \cap B(x, r))$$

and

$$\overline{\dim}_{\mathcal{B},\text{loc}}(x, K) = \lim_{r \searrow 0} \overline{\dim}_{\mathcal{B}}(K \cap B(x, r)),$$

respectively. We can now state the result from [6] giving the  $q$ -Rényi dimensions of a typical measure for  $q \in (0, \infty)$ .

**Theorem A.** [6] *Let  $K$  be a compact subset of  $\mathbb{R}^d$ . Write*

$$\underline{s} = \inf_{x \in K} \underline{\dim}_{\mathcal{B},\text{loc}}(x, K),$$

$$\bar{s} = \inf_{x \in K} \overline{\dim}_{\mathcal{B},\text{loc}}(x, K),$$

$$s = \overline{\dim}_{\mathcal{B}}(K).$$

*Observe that  $\underline{s} \leq \bar{s} \leq s$ . Assume that  $\underline{s} = \bar{s} = s$  (this condition is clearly satisfied if, for example,  $K$  is the closure of an open and bounded set or if  $K$  is a self-similar set satisfying the open set condition).*

(1) *For all measures  $\mu \in \mathcal{P}(K)$  we have*

$$0 \leq \underline{D}_{\mu}(q) \leq \overline{D}_{\mu}(q) \leq s$$

*for all  $q \in (1, \infty)$ .*

(2) A typical measure  $\mu \in \mathcal{P}(K)$  satisfies the following

$$\underline{D}_\mu(q) = 0,$$

$$\overline{D}_\mu(q) = s,$$

for all  $q \in (1, \infty)$ .

The purpose of the paper is to show that this result extends to the following two limiting cases, namely, for  $q = 1$  and for  $q = \infty$ . However, we first give a general result providing the relationship between the  $q$ -Rényi dimensions for different values of  $q \in [0, \infty]$ . This result will be useful later.

**Proposition 1.** Let  $K$  be a compact subset of  $\mathbb{R}^d$  and write  $s = \overline{\dim}_B(K)$ . For all measures  $\mu \in \mathcal{P}(K)$  and all  $q \in (1, \infty)$  we have

$$0 \leq \underline{D}_\mu(\infty) \leq \underline{D}_\mu(q) \leq \underline{D}_\mu(1) \leq s,$$

and

$$0 \leq \overline{D}_\mu(\infty) \leq \overline{D}_\mu(q) \leq \overline{D}_\mu(1) \leq s.$$

**Proof.** It is clear that  $0 \leq \underline{D}_\mu(\infty) \leq \underline{D}_\mu(q)$  and  $0 \leq \overline{D}_\mu(\infty) \leq \overline{D}_\mu(q)$ , and it follows easily from Jensen's inequality that  $\underline{D}_\mu(q) \leq \underline{D}_\mu(1)$  and  $\overline{D}_\mu(q) \leq \overline{D}_\mu(1)$ . Hence, it suffices to show that  $\overline{D}_\mu(1) \leq s$ . For a positive real number  $r > 0$ , let  $N_r(K)$  denote the smallest number of balls of radius equal to  $r$  that is needed to cover the set  $K$ . Then  $s = \overline{\dim}_B(K) = \limsup_{r \searrow 0} \frac{\log N_r(K)}{-\log r}$ , cf. [1]. Fix  $r > 0$ . For brevity write  $N = N_r(E)$ . We can thus choose balls  $B(x_1, r), \dots, B(x_N, r)$  such that  $K \subseteq \bigcup_i B(x_i, r)$ . Put  $K_1 = B(x_1, r)$  and  $K_i = B(x_i, r) \setminus \bigcup_{j=1}^{i-1} B(x_j, r)$  for  $i = 2, \dots, N$ . Next observe that if  $x \in K_i$ , then  $K_i \subseteq B(x, 2r)$ . We conclude from this and Jensen's inequality applied to the function  $\Phi: (0, \infty) \rightarrow \mathbb{R}$  defined by  $\Phi(t) = t \log t$  that

$$\begin{aligned} \int \mu(B(x, 2r)) d\mu(x) &= \sum_i \int_{K_i \cap K} \log \mu(B(x, 2r)) d\mu(x) \\ &\geq \sum_i \int_{K_i \cap K} \log \mu(K_i \cap K) d\mu(x) \\ &= \sum_i \mu(K_i \cap K) \log \mu(K_i \cap K) \\ &= N \sum_i \frac{1}{N} \Phi(\mu(K_i \cap K)) \\ &\geq N \Phi\left(\sum_i \frac{1}{N} \mu(K_i \cap K)\right) \\ &= N \Phi\left(\frac{1}{N}\right) \\ &= -\log N. \end{aligned} \tag{1.5}$$

The desired conclusion now follows from (1.5) by dividing by  $\log r$ .  $\square$

We will now state the main results in the paper extending the results in Theorem A to the cases  $q = 1$  and  $q = \infty$ .

**Theorem 2** (The case:  $q = \infty$ ). Let  $K$  be a compact subset of  $\mathbb{R}^d$ . Let  $\underline{s}$ ,  $\bar{s}$  and  $s$  be defined as in Theorem A.

(1) For all measures  $\mu \in \mathcal{P}(K)$  we have

$$0 \leq \underline{D}_\mu(\infty) \leq \bar{D}_\mu(\infty) \leq s.$$

(2) A typical measure  $\mu \in \mathcal{P}(K)$  satisfies the following

$$\begin{aligned} \underline{D}_\mu(\infty) &= 0, \\ \underline{s} &\leq \bar{D}_\mu(\infty) \leq \bar{s}. \end{aligned}$$

In particular, if  $\underline{s} = \bar{s} = s$  (this condition is clearly satisfied if, for example,  $K$  is the closure of an open and bounded set or if  $K$  is a self-similar set satisfying the open set condition), then a typical measure  $\mu \in \mathcal{P}(K)$  satisfies the following

$$\begin{aligned} \underline{D}_\mu(\infty) &= 0, \\ \bar{D}_\mu(\infty) &= s. \end{aligned}$$

**Theorem 3** (The case:  $q = 1$ ). Let  $K$  be a compact subset of  $\mathbb{R}^d$ . Let  $\underline{s}$ ,  $\bar{s}$  and  $s$  be defined as in Theorem A.

(1) For all measures  $\mu \in \mathcal{P}(K)$  we have

$$0 \leq \underline{D}_\mu(1) \leq \bar{D}_\mu(1) \leq s.$$

(2) A typical measure  $\mu \in \mathcal{P}(K)$  satisfies the following

$$\underline{s} \leq \bar{D}_\mu(1) \leq s.$$

In particular, if  $\underline{s} = \bar{s} = s$  (this condition is clearly satisfied if, for example,  $K$  is the closure of an open and bounded set or if  $K$  is a self-similar set satisfying the open set condition), then a typical measure  $\mu \in \mathcal{P}(K)$  satisfies the following

$$\bar{D}_\mu(1) = s.$$

Observe that part (1) of Theorem 2 follows immediately from Proposition 1, and that Theorem 3 follows immediately from Proposition 1 and Theorem 2. Part (2) of Theorem 2 is proved in Section 3.

Note that the second half of part (2) of Theorem A follows immediately from Proposition 1 and Theorem 2, namely, since  $\bar{D}_\mu(\infty) \leq \bar{D}_\mu(q) \leq s$  for all  $\mu \in \mathcal{P}(K)$  and all  $q \in (1, \infty)$ , we conclude from Theorem 2 that if  $\underline{s} = \bar{s} = s$ , then a typical measure  $\mu \in \mathcal{P}(K)$  satisfies the following

$$\bar{D}_\mu(q) = s$$

for all  $q \in (1, \infty)$ . This provides an alternative proof of the second half of part (2) of Theorem A.

Comparing the statements in part (1) and part (2) of Theorem A, Theorem 2 and Theorem 3, we see that a typical measure  $\mu$  is as irregular as possible: for all  $q \in (1, \infty]$ , the lower  $q$ -Rényi

dimension  $\underline{D}_\mu(q)$  attains the smallest possible value and for all  $q \in [1, \infty]$  the upper  $q$ -Rényi dimension  $\overline{D}_\mu(q)$  attains the largest possible value. In particular, Theorems A, 2 and 3 show that a typical measure fails to satisfy part (1.3) of the Multifractal Formalism in a very spectacular way.

The typical behaviour of various other quantities related to multifractal analysis has also been studied. In particular, the local dimension  $\dim_{\text{loc}}(x; \mu)$  of a typical measure has been studied by Haase [3] and investigated further by Genyuk [2].

## 2. Proof of part (2) of Theorem 2

Write

$$\begin{aligned}\Gamma &= \{\mu \in \mathcal{P}(K) \mid \underline{D}_\mu(\infty) = 0\}, \\ \Delta^u &= \{\mu \in \mathcal{P}(K) \mid \underline{s} \leq \overline{D}_\mu(\infty)\}, \\ \Delta^l &= \{\mu \in \mathcal{P}(K) \mid \overline{D}_\mu(\infty) \leq \bar{s}\}.\end{aligned}\tag{2.1}$$

We must prove that the three sets  $\Gamma$ ,  $\Delta^u$  and  $\Delta^l$  are residual. In Section 2.1 we prove that the set  $\Gamma$  is residual, in Section 2.2 we prove that the set  $\Delta^u$  is residual, and finally in Section 2.3 we prove that the set  $\Delta^l$  is residual.

It is well known (cf., for example, [7, p. 51, Theorem 6.8]) that the weak topology on  $\mathcal{P}(K)$  is induced by the metric  $L$  on  $\mathcal{P}(K)$  defined as follows. Let  $\text{Lip}(K)$  denote the family of Lipschitz functions  $f: K \rightarrow \mathbb{R}$  with  $|f| \leq 1$  and  $\text{Lip}(f) \leq 1$  where  $\text{Lip}(f)$  denotes the Lipschitz constant of  $f$ . The metric  $L$  is now defined by

$$L(\mu, \nu) = \sup_{f \in \text{Lip}(K)} \left| \int f d\mu - \int f d\nu \right|$$

for  $\mu, \nu \in \mathcal{P}(K)$ . We will always equip  $\mathcal{P}(K)$  with the metric  $L$  and all balls in  $\mathcal{P}(K)$  will be with respect to the metric  $L$ , i.e. if  $\mu \in \mathcal{P}(K)$  and  $r > 0$  we will write  $B(\mu, r) = \{\nu \in \mathcal{P}(K) \mid L(\mu, \nu) < r\}$  for the ball with centre at  $\mu$  and radius equal to  $r$ . For  $x \in K$  and  $r > 0$ , define  $f_{x,r}: K \rightarrow \mathbb{R}$  by

$$f_{x,r}(t) = \begin{cases} r & \text{if } |x - t| \leq r, \\ -|t - x| + 2r & \text{if } r < |x - t| < 2r, \\ 0 & \text{if } 2r \leq |x - t|. \end{cases}\tag{2.2}$$

Observe that if  $r \leq 1$ , then  $f_{x,r}$  is Lipschitz with  $|f_{x,r}| \leq 1$  and  $\text{Lip}(f_{x,r}) = 1$ . In particular, this implies that if  $r \leq 1$ , then

$$\left| \int f_{x,r} d\mu - \int f_{x,r} d\nu \right| \leq L(\mu, \nu)\tag{2.3}$$

for all  $\mu, \nu \in \mathcal{P}(K)$ . This inequality will be used frequently in Sections 2.1–2.3.

Finally, for a probability measure  $\mu$  and  $r > 0$ , write

$$I_\mu(\infty; r) = \sup_{x \in \text{supp } \mu} \mu(B(x, r)).\tag{2.4}$$

## 2.1. The set $\Gamma$ is residual

In this section we prove that the set  $\Gamma$  is residual. It clearly suffices to construct a set  $M \subseteq \mathcal{P}(K)$  satisfying the following three conditions:

- (1)  $M \subseteq \Gamma$ ;
- (2)  $M$  is dense in  $\mathcal{P}(K)$ ;
- (3)  $M$  is  $\mathcal{G}_\delta$ .

For a positive integer write

$$A_n = \left\{ \lambda \in \mathcal{P}(K) \mid \lambda(\{x_0\}) \geq \frac{1}{n} \text{ for some } x_0 \in K \right\}.$$

Next put

$$G_n = \bigcup_{\lambda \in A_n} B\left(\lambda, \frac{1}{9n^{n+1}}\right),$$

and define the set  $M \subseteq \mathcal{P}(K)$  by

$$M = \bigcap_m \bigcup_{n \geq m} G_n.$$

Below we show that the set  $M$  has the following three properties: (1)  $M \subseteq \Gamma$ , (2)  $M$  is dense in  $\mathcal{P}(K)$ , and (3)  $M$  is  $\mathcal{G}_\delta$ . The set  $M$  is clearly  $\mathcal{G}_\delta$ , and it thus suffices to show that  $M \subseteq \Gamma$  and that  $M$  is dense in  $\mathcal{P}(K)$ . This is done in Proposition 2.1.1 and Lemma 2.1.2.

**Proposition 2.1.1.** *We have  $M \subseteq \Gamma$ .*

**Proof.** Let  $\mu \in M$  and fix a positive integer  $m$ . Since  $\mu \in M$ , there exists  $n \geq m$  and a measure  $\lambda \in A_n$  such that  $L(\mu, \lambda) \leq \frac{1}{9n^{n+1}}$ . Also, since  $\lambda \in A_n$ , we can find a point  $x_0 \in K$  with  $\lambda(\{x_0\}) \geq \frac{1}{n}$ . For brevity write  $r_n = \frac{1}{n^n}$ . Now observe that for all  $x \in B(x_0, \frac{r_n}{3})$  we have (using (2.3))

$$\begin{aligned} \mu(B(x, r_n)) &= \int 1_{B(x, r_n)} d\mu \\ &\geq \int \frac{f_{x_0, \frac{r_n}{3}}}{\frac{r_n}{3}} d\mu \\ &\geq \frac{3}{r_n} \left( -L(\lambda, \mu) + \int f_{x_0, \frac{r_n}{3}} d\lambda \right) \\ &\geq \frac{3}{r_n} \left( -\frac{1}{9n^{n+1}} + f_{x_0, \frac{r_n}{3}}(x_0) \lambda(\{x_0\}) \right) \\ &\geq \frac{3}{r_n} \left( -\frac{1}{9n^{n+1}} + \frac{r_n}{3} \frac{1}{n} \right) = \frac{2}{3n} \end{aligned} \tag{2.5}$$

and (using (2.3) once more)

$$\begin{aligned}
\mu\left(B\left(x_0, \frac{r_n}{3}\right)\right) &= \int 1_{B(x_0, \frac{r_n}{3})} d\mu \\
&\geq \int \frac{f_{x_0, \frac{r_n}{6}}}{\frac{r_n}{6}} d\mu \\
&\geq \frac{6}{r_n} \left( -L(\lambda, \mu) + \int f_{x_0, \frac{r_n}{6}} d\lambda \right) \\
&\geq \frac{6}{r_n} \left( -\frac{1}{9n^{n+1}} + f_{x_0, \frac{r_n}{6}}(x_0) \lambda(\{x_0\}) \right) \\
&\geq \frac{6}{r_n} \left( -\frac{1}{9n^{n+1}} + \frac{r_n}{6} \frac{1}{n} \right) = \frac{6}{27n}.
\end{aligned}$$

In particular, this implies that  $\mu(B(x_0, \frac{r_n}{3})) > 0$ , and we therefore conclude that there exists  $y_n \in B(x_0, \frac{r_n}{3}) \cap \text{supp } \mu$ . Since  $y_n \in B(x_0, \frac{r_n}{3})$  it follows from (2.5) that  $\mu(B(y_n, r_n)) \geq \frac{2}{3n}$ . Hence

$$I_\mu(\infty; r_n) = \sup_{x \in \text{supp } \mu} \mu(B(x, r_n)) \geq \mu(B(y_n, r_n)) \geq \frac{2}{3n}.$$

This implies that

$$\begin{aligned}
\underline{D}_\mu(\infty) &= \liminf_{r \searrow 0} \frac{\log I_\mu(\infty; r)}{\log r} \\
&\leq \liminf_n \frac{\log I_\mu(\infty; r_n)}{\log r_n} \\
&\leq \liminf_n \frac{\log \frac{2}{3n}}{-n \log n} \\
&= 0.
\end{aligned}$$

This completes the proof of Proposition 2.1.1.  $\square$

**Lemma 2.1.2.** *The set  $M$  is dense in  $\mathcal{P}(K)$ .*

**Proof.** Since  $\mathcal{P}(K)$  is a complete metric space (because  $K$  is compact) and each set  $\bigcup_{n \geq m} G_n$  is open, it suffices (by Baire's Theorem) to show that  $\bigcup_{n \geq m} G_n$  is dense for all  $m$ . In order to show that  $\bigcup_{n \geq m} G_n$  is dense it suffices to show that the subset  $\bigcup_{n \geq m} A_n$  is dense for all  $m$ . Therefore fix a positive integer  $m$ . Let  $\mu \in \mathcal{P}(K)$  and  $0 < \varepsilon \leq 1$ . Pick any  $x_0 \in K$ . Next, choose a positive integer  $n_0 \geq m$  with  $\frac{1}{n_0} \leq \frac{\varepsilon}{2}$  and put  $\lambda = \frac{\varepsilon}{2} \delta_{x_0} + (1 - \frac{\varepsilon}{2})\mu$ . Then  $\lambda(\{x_0\}) \geq \frac{\varepsilon}{2} \geq \frac{1}{n_0}$ , whence  $\lambda \in A_{n_0} \subseteq \bigcup_{n \geq m} A_n$ . Also,  $L(\mu, \lambda) = \sup_{f \in \text{Lip}(K)} |\int f d\mu - \int f d\lambda| = \sup_{f \in \text{Lip}(K)} \frac{\varepsilon}{2} |\int f d\mu - f(x_0)| \leq \sup_{f \in \text{Lip}(K)} \varepsilon = \varepsilon$ . This shows that  $\bigcup_{n \geq m} A_n$  is dense for all  $m$ .  $\square$

## 2.2. The set $\Delta^u$ is residual

In this section we prove that the set  $\Delta^u$  is residual. For a real number  $t$  write

$$\Delta_t^u = \{\mu \in \mathcal{P}(K) \mid t \leq \overline{D}_\mu(\infty)\}.$$

Since

$$\Delta^u = \bigcap_{\substack{t \in \mathbb{Q} \\ t < s}} \Delta_t^u,$$



it clearly suffices to prove that the set  $\Delta_t^u$  is residual for each rational number  $t$  with  $t < \underline{s}$ . Therefore fix a rational number  $t$  with  $t < \underline{s}$ . To prove that the set  $\Delta_t^u$  is residual it clearly suffices to construct a set  $M^u \subseteq \mathcal{P}(K)$  satisfying the following three conditions:

- (1)  $M^u \subseteq \Delta_t^u$ ;
- (2)  $M^u$  is dense in  $\mathcal{P}(K)$ ;
- (3)  $M^u$  is  $\mathcal{G}_\delta$ .

**Lemma 2.2.1.** Assume that  $x_0 \in K$ ,  $r_0 > 0$  and  $t \geq 0$  satisfy

$$t < \underline{\dim}_B(K \cap B(x_0, r_0)).$$

Then there exists  $c > 0$  such that for each  $r > 0$  there exists a measure  $\mu \in \mathcal{P}(K)$  with

- (1)  $\text{supp } \mu \subseteq K \cap B(x_0, r_0)$ ;
- (2) for all  $x \in K$  we have  $\mu(B(x, r)) \leq cr^t$ .

**Proof.** For  $r > 0$ , let  $M_r(K \cap B(x_0, r_0))$  denote the largest number of pairwise disjoint balls of radius  $r$  with centres in  $K \cap B(x_0, r_0)$ . Then  $\underline{\dim}_B(K \cap B(x_0, r_0)) = \liminf_{r \searrow 0} \frac{\log M_r(K \cap B(x_0, r_0))}{-\log r}$ , cf. [1]. We can thus find  $0 < \delta \leq 1$  such that  $\frac{\log M_r(K \cap B(x_0, r_0))}{-\log r} > t$  for all  $0 < r \leq \delta$ , whence

$$M_r(K \cap B(x_0, r_0)) > r^{-t} \quad (2.6)$$

for all  $0 < r \leq \delta$ . Now put  $c = \frac{1}{\delta^t} \geq 1$ . We must prove that for each  $r > 0$  there exists a measure  $\mu \in \mathcal{P}(K)$  satisfying conditions (1) and (2). Therefore fix  $r > 0$ . We divide the proof into two cases.

*Case 1:*  $\delta < r$ . Pick any  $\mu \in \mathcal{P}(K)$  with  $\text{supp } \mu \subseteq K \cap B(x_0, r_0)$ . (For example, we may put  $\mu = \delta_{x_0}$ .) For all  $x \in K$  we clearly have  $\mu(B(x, r)) \leq 1 = c\delta^t < cr^t$ .

*Case 2:*  $0 < r \leq \delta$ . For brevity write  $M = M_r(K \cap B(x_0, r_0))$ . By definition of  $M$  there exist  $M$  pairwise disjoint balls  $B(x_1, r), \dots, B(x_M, r)$  with centres  $x_1, \dots, x_M$  in  $K \cap B(x_0, r_0)$ . Now put  $\mu = \frac{1}{M} \sum_{i=1}^M \delta_{x_i}$ . Then clearly  $\text{supp } \mu \subseteq K \cap B(x_0, r_0)$ . Next, let  $x \in K$  and observe that the ball  $B(x, r)$  can at most contain one of the  $x_i$ 's. Indeed, otherwise there exist two distinct indices  $i$  and  $j$  such that  $x_i, x_j \in B(x, r)$ , whence  $x \in B(x_i, r) \cap B(x_j, r)$ , contradicting the fact that the balls  $B(x_1, r), \dots, B(x_M, r)$  are pairwise disjoint. Since  $r \leq \delta$  and the ball  $B(x, r)$  contains at most one of the  $x_i$ 's, we conclude from (2.6) that

$$\mu(B(x, r)) \leq \frac{1}{M} = \frac{1}{M_r(K \cap B(x_0, r_0))} < r^t \leq cr^t.$$

This completes the proof of Lemma 2.2.1.  $\square$

Let  $(x_n)_n$  be a dense sequence in  $K$ . Fix  $n$  and  $i = 1, \dots, n$ . Since

$$t < \underline{s} = \inf_{x \in K} \underline{\dim}_{B, \text{loc}}(x, K) \leq \underline{\dim}_B\left(K \cap B\left(x_i, \frac{1}{n}\right)\right),$$

it follows from Lemma 2.2.1 that there exists a constant  $c_{n,i}$  such that for all  $r > 0$  there exists a measure  $\mu \in \mathcal{P}(K)$  with

- (1)  $\text{supp } \mu \subseteq K \cap B(x_i, \frac{1}{n})$ ;  
 (2) for all  $x \in K$  we have  $\mu(B(x, r)) \leq c_{n,i} r^t$ .

Now put  $c_n = \max(2^t c_{n,1}, \dots, 2^t c_{n,n}, n)$  and  $r_n = \frac{1}{e^{c_n}}$ . We can thus choose a measure  $\mu_{n,i} \in \mathcal{P}(K)$  with

- (1)  $\text{supp } \mu_{n,i} \subseteq K \cap B(x_i, \frac{1}{n})$ ;  
 (2) for all  $x \in K$  we have  $\mu_{n,i}(B(x, 2r_n)) \leq c_{n,i}(2r_n)^t$ .

For a positive integer  $n$  write

$$\Lambda_n^u = \left\{ \sum_{i=1}^n p_i \mu_{n,i} \mid p_i \geq 0, \sum_{i=1}^n p_i = 1 \right\}.$$

Next put

$$G_n^u = \bigcup_{\lambda \in \Lambda_n^u} B(\lambda, r_n^{t+1}),$$

and define the set  $M^u \subseteq \mathcal{P}(K)$  by

$$M^u = \bigcap_m \bigcup_{n \geq m} G_n^u.$$

Below we show that the set  $M^u$  has the following three properties: (1)  $M^u \subseteq \Delta_t^u$ , (2)  $M^u$  is dense in  $\mathcal{P}(K)$ , and (3)  $M^u$  is  $\mathcal{G}_\delta$ . The set  $M^u$  is clearly  $\mathcal{G}_\delta$ , and it thus suffices to show that  $M^u \subseteq \Delta_t^u$  and that  $M^u$  is dense in  $\mathcal{P}(K)$ . This is done in Proposition 2.2.2 and Lemma 2.2.4.

**Proposition 2.2.2.** *We have  $M^u \subseteq \Delta_t^u$ .*

**Proof.** Let  $\mu \in M^u$  and fix a positive integer  $m$ . Since  $\mu \in M^u$ , there exists  $n \geq m$  and a measure  $\lambda \in \Lambda_n^u$  such that  $L(\mu, \lambda) \leq r_n^{t+1}$ . Also, since  $\lambda \in \Lambda_n^u$ , we can find  $p_1, \dots, p_n$  with  $p_i \geq 0$  and  $\lambda = \sum_i p_i \mu_{n,i}$ . Now observe that for all  $x \in K$  we have (using (2.3))

$$\begin{aligned} \mu(B(x, r_n)) &= \int 1_{B(x, r_n)} d\mu \\ &\leq \int \frac{f_{x, r_n}}{r_n} d\mu \\ &\leq \frac{1}{r_n} \left( L(\mu, \lambda) + \int f_{x, r_n} d\lambda \right) \\ &\leq \frac{1}{r_n} (L(\mu, \lambda) + r_n \lambda(B(x, 2r_n))) \\ &\leq \frac{1}{r_n} \left( r_n^{t+1} + r_n \sum_i p_i \mu_{n,i}(B(x, 2r_n)) \right) \\ &\leq \frac{1}{r_n} \left( r_n^{t+1} + r_n \sum_i p_i c_{n,i} (2r_n)^t \right) \\ &\leq \frac{1}{r_n} \left( r_n^{t+1} + r_n \sum_i p_i c_n r_n^t \right) = (1 + c_n) r_n^t. \end{aligned}$$

This implies that

$$I_\mu(\infty; r_n) = \sup_{x \in \text{supp } \mu} \mu(B(x, r_n)) \leq (1 + c_n)r_n^t.$$

Hence

$$\begin{aligned} \bar{D}_\mu(\infty) &= \limsup_{r \searrow 0} \frac{\log I_\mu(\infty; r)}{-\log r} \\ &\geq \limsup_n \frac{\log I_\mu(\infty; r_n)}{\log r_n} \\ &\geq \limsup_n \frac{\log(1 + c_n) + t \log r_n}{\log r_n} \\ &= t. \end{aligned}$$

This completes the proof of Proposition 2.2.2.  $\square$

**Lemma 2.2.3.** *Let  $F \subseteq \mathbb{R}^d$  be a bounded Borel set and  $r > 0$ . Then there exists finitely many pairwise disjoint Borel sets  $F_1, \dots, F_N$  with  $\text{diam } F_j \leq r$  such that  $F \subseteq \bigcup_j F_j$ , and such that for each  $j$ , there exists an  $x_j \in F$  satisfying*

$$B\left(x_j, \frac{r}{4}\right) \subseteq F_j.$$

**Proof.** First construct a sequence of balls  $B(x_1, \frac{r}{2}), B(x_2, \frac{r}{2}), \dots$  such that  $x \in F$  and  $|x_i - x_j| > \frac{r}{2}$  for all  $i \neq j$ . Because  $F$  is totally bounded this process must terminate at some finite stage, giving balls  $B(x_1, \frac{r}{2}), B(x_2, \frac{r}{2}), \dots, B(x_N, \frac{r}{2})$  such that any  $x \in F$  must satisfy  $\min_j |x - x_j| \leq \frac{r}{2}$  (and consequently  $F \subseteq \bigcup_{j=1}^N B(x_j, \frac{r}{2})$ ). Note that the smaller balls  $B(x_1, \frac{r}{4}), B(x_2, \frac{r}{4}), \dots, B(x_N, \frac{r}{4})$  are pairwise disjoint. Now set

$$\begin{aligned} F_1 &= B\left(x_1, \frac{r}{2}\right) \setminus \bigcup_{i=2}^N B\left(x_i, \frac{r}{4}\right), \\ F_j &= B\left(x_j, \frac{r}{2}\right) \setminus \left( \bigcup_{i=1}^{j-1} F_i \cup \bigcup_{i=j+1}^N B\left(x_i, \frac{r}{4}\right) \right) \quad \text{for } j = 2, \dots, N-1, \\ F_N &= B\left(x_N, \frac{r}{2}\right) \setminus \bigcup_{i=1}^{N-1} F_i. \end{aligned}$$

It is clear that the sets  $F_1, F_2, \dots, F_N$  are pairwise disjoint, and since  $B(x_1, \frac{r}{4}), B(x_2, \frac{r}{4}), \dots, B(x_N, \frac{r}{4})$  are pairwise disjoint we conclude that  $B(x_j, \frac{r}{4}) \subseteq F_j$  and  $F \subseteq \bigcup_j F_j$ .  $\square$

**Lemma 2.2.4.** *The set  $M^u$  is dense in  $\mathcal{P}(K)$ .*

**Proof.** Since  $\mathcal{P}(K)$  is a complete metric space (because  $K$  is compact) and each set  $\bigcup_{k \geq m} G_k^u$  is open, it suffices (by Baire's Theorem) to show that  $\bigcup_{k \geq m} G_k^u$  is dense for all  $m$ . In order to show that  $\bigcup_{k \geq m} G_k^u$  is dense it suffices to show that the subset  $\bigcup_{k \geq m} \Lambda_k^u$  is dense for all  $m$ . Therefore fix a positive integer  $m$ . Let  $\mu \in \mathcal{P}(K)$  and  $0 < \varepsilon \leq 1$ . According to Lemma 2.2.3 we

may choose finitely many pairwise disjoint Borel sets  $K_1, \dots, K_N$  with  $\text{diam } K_j \leq \varepsilon$  such that  $K \subseteq \bigcup_j K_j$ , and such that for each  $j$  there exists an  $y_j \in K$  satisfying

$$B\left(y_j, \frac{\varepsilon}{4}\right) \subseteq K_j.$$

Since the sequence  $(x_k)_k$  is dense in  $K$  we can also choose a positive integer  $n \geq m$  such that  $\frac{1}{n} \leq \frac{\varepsilon}{8}$  and  $\{x_1, \dots, x_n\} \cap B(y_j, \frac{\varepsilon}{8}) \neq \emptyset$  for all  $j$ . Hence, for each  $j = 1, \dots, N$  we can pick a (not necessarily unique)  $i(j)$  with

$$x_{i(j)} \in B\left(y_j, \frac{\varepsilon}{8}\right).$$

Now put

$$p_i = \begin{cases} \mu(K \cap K_j) & \text{if } i = i(j) \text{ for some } j = 1, \dots, N; \\ 0 & \text{if } i \neq i(j) \text{ for all } j = 1, \dots, N. \end{cases}$$

Finally, write

$$\lambda = \sum_i p_i \mu_{n,i}.$$

We will now show that  $\lambda \in \bigcup_{k \geq m} A_k^u$  and that  $L(\mu, \lambda) \leq \varepsilon$ . Indeed, we clearly have that  $\lambda \in A_n^u \subseteq \bigcup_{k \geq m} A_k^u$ . Next, we prove that  $L(\mu, \lambda) \leq \varepsilon$ . We have

$$\begin{aligned} L(\mu, \lambda) &= \sup_{f \in \text{Lip}(K)} \left| \int f d\mu - \int f d\lambda \right| \\ &\leq \sup_{f \in \text{Lip}(K)} \sum_j \left| \int_{K \cap K_j} f d\mu - \int_{K \cap K_j} f d\lambda \right|. \end{aligned} \quad (2.7)$$

First, observe that if  $f : K \rightarrow \mathbb{R}$  is a real valued function with  $\text{Lip}(f) \leq 1$  and  $|f| \leq 1$ , then

$$\mu(K \cap K_j) \inf_{x \in K \cap K_j} f(x) \leq \int_{K \cap K_j} f d\mu \leq \mu(K \cap K_j) \sup_{x \in K \cap K_j} f(x). \quad (2.8)$$

Next, observe that since  $\text{supp } \mu_{n,i(j)} \subseteq K \cap B(x_{i(j)}, \frac{1}{n}) \subseteq K \cap B(x_{i(j)}, \frac{\varepsilon}{8}) \subseteq K \cap B(y_j, \frac{\varepsilon}{4}) \subseteq K \cap K_j$  and the sets  $K_1, \dots, K_N$  are pairwise disjoint, we have

$$\int_{K \cap K_j} f d\lambda = p_{i(j)} \int_{K \cap K_j} f d\mu_{n,i(j)}.$$

It follows from this that

$$\int_{K \cap K_j} f d\lambda \leq p_{i(j)} \mu_{n,i(j)}(K \cap K_j) \sup_{x \in K \cap K_j} f(x) = \mu(K \cap K_j) \sup_{x \in K \cap K_j} f(x), \quad (2.9)$$

and that

$$\int_{K \cap K_j} f d\lambda \geq p_{i(j)} \mu_{n,i(j)}(K \cap K_j) \inf_{x \in K \cap K_j} f(x) = \mu(K \cap K_j) \inf_{x \in K \cap K_j} f(x). \quad (2.10)$$

Finally combining (2.8)–(2.10) show that

$$\begin{aligned} \left| \int_{K \cap K_j} f d\mu - \int_{K \cap K_j} f d\lambda \right| &\leq \mu(K \cap K_j) \left( \sup_{x \in K \cap K_j} f(x) - \inf_{x \in K \cap K_j} f(x) \right) \\ &\leq \mu(K \cap K_j) \operatorname{diam}(K \cap K_j). \end{aligned} \quad (2.11)$$

It now follows from (2.7) and (2.11) that

$$\begin{aligned} L(\mu, \lambda) &\leq \sup_{f \in \operatorname{Lip}(K)} \sum_j \mu(K \cap K_j) \operatorname{diam}(K \cap K_j) \\ &\leq \varepsilon \sum_j \mu(K \cap K_j) \\ &= \varepsilon \mu \left( K \cap \bigcup_j K_j \right) \\ &= \varepsilon. \end{aligned}$$

This completes the proof.  $\square$

### 2.3. The set $\Delta^1$ is residual

In this section we prove that the set  $\Delta^1$  is residual. For a real number  $t$  write

$$\Delta_t^1 = \{ \mu \in \mathcal{P}(K) \mid \overline{D}_\mu(\infty) \leq \bar{s} \}.$$

Since

$$\Delta^1 = \bigcap_{\substack{t \in \mathbb{Q} \\ \bar{s} < t}} \Delta_t^1,$$

it clearly suffices to prove that the set  $\Delta_t^1$  is residual for each rational number  $t$  with  $\bar{s} < t$ . Therefore fix a rational number  $t$  with  $\bar{s} < t$ . To prove that the set  $\Delta_t^1$  is residual it clearly suffices to construct a set  $M^1 \subseteq \mathcal{P}(K)$  satisfying the following three conditions:

- (1)  $M^1 \subseteq \Delta_t^1$ ;
- (2)  $M^1$  is dense in  $\mathcal{P}(K)$ ;
- (3)  $M^1$  is  $\mathcal{G}_\delta$ .

Put

$$\begin{aligned} \Lambda^1 = \left\{ \lambda \in \mathcal{P}(K) \mid \text{there exists } x_0 \in K \text{ and } r_0 > 0 \text{ such that} \right. \\ \left. \overline{\dim}_B(K \cap B(x_0, r_0)) < t \text{ and } \lambda \left( B \left( x_0, \frac{r_0}{2} \right) \right) > 0 \right\}. \end{aligned}$$

Hence, for  $\lambda \in \Lambda^1$  there exist  $x_0 \in K$  and  $r_0 > 0$  such that  $\overline{\dim}_B(K \cap B(x_0, r_0)) < t$  and  $\lambda(B(x_0, \frac{r_0}{2})) > 0$ ; we now write  $r_\lambda = \frac{r_0}{4} \lambda(B(x_0, \frac{r_0}{2}))$ . Put

$$M^1 = \bigcup_{\lambda \in \Lambda^1} B(\lambda, r_\lambda).$$

Below we show that the set  $M^1$  has the following three properties: (1)  $M^1 \subseteq \Delta_t^1$ , (2)  $M^1$  is dense in  $\mathcal{P}(K)$ , and (3)  $M^1$  is  $\mathcal{G}_\delta$ . The set  $M^1$  is clearly  $\mathcal{G}_\delta$ , and it thus suffices to show that  $M^1 \subseteq \Delta_t^1$  and that  $M^1$  is dense in  $\mathcal{P}(K)$ . This is done in Propositions 2.3.2 and 2.3.3.

**Lemma 2.3.1.** *Let  $\mu \in \mathcal{P}(K)$  and  $E \subseteq K$  with  $\mu(E) > 0$ . Then*

$$\overline{D}_\mu(\infty) \leq \overline{\dim}_B(E).$$

**Proof.** For a positive real number  $r > 0$ , let  $N_r(E)$  denote the smallest number of balls of radius equal to  $r$  that is needed to cover the set  $E$ . Then  $\overline{\dim}_B(E) = \limsup_{r \searrow 0} \frac{\log N_r(E)}{-\log r}$ , cf. [1]. We will now show that

$$I_\mu(\infty; 2r) \geq \mu(E) \frac{1}{N_r(E)} \quad (2.12)$$

for all  $r > 0$ . Therefore fix  $r > 0$ . For brevity write  $N = N_r(E)$ . We can thus choose balls  $B(x_1, r), \dots, B(x_N, r)$  such that  $E \subseteq \bigcup_i B(x_i, r)$ . Put  $E_1 = B(x_1, r)$  and  $E_i = B(x_i, r) \setminus \bigcup_{j=1}^{i-1} B(x_j, r)$  for  $i = 2, \dots, N$ . Next observe that if  $x \in E_i$ , then  $E_i \subseteq B(x, 2r)$ . We conclude from this that

$$\begin{aligned} I_\mu(\infty; 2r) &= \sup_{x \in \text{supp } \mu} \mu(B(x, 2r)) \\ &= \max_i \sup_{x \in \text{supp } \mu \cap E_i \cap E} \mu(B(x, 2r)) \\ &= \max_i \sup_{x \in \text{supp } \mu \cap E_i \cap E} \mu(E_i \cap E) \\ &= \max_i \mu(E_i \cap E) \\ &\geq \frac{1}{N} \sum_i \mu(E_i \cap E) \\ &= \frac{1}{N} \mu\left(\bigcup_i (E_i \cap E)\right) \\ &= \frac{1}{N} \mu(E). \end{aligned}$$

This completes the proof of (2.12).

Since  $\mu(E) > 0$ , the desired conclusion now follows from (2.12) by taking logarithms and dividing by  $\log r$ .  $\square$

**Proposition 2.3.2.** *We have  $M^1 \subseteq \Delta_t^1$ .*

**Proof.** Let  $\mu \in M^1$ . We can thus choose  $\lambda \in \Lambda^1$  such that  $L(\mu, \lambda) \leq r_\lambda$  where  $r_\lambda = \frac{r_0}{4} \lambda(B(x_0, \frac{r_0}{2}))$  for some  $x_0 \in K$  and  $r_0 > 0$  with  $\overline{\dim}_B(K \cap B(x_0, r_0)) < t$  and  $\lambda(B(x_0, \frac{r_0}{2})) > 0$ . It now follows that (using (2.3))

$$\begin{aligned} \mu(K \cap B(x_0, r_0)) &= \int 1_{B(x_0, r_0)} d\mu \\ &\geq \int \frac{f_{x_0, \frac{r_0}{2}}}{\frac{r_0}{2}} d\mu \end{aligned}$$

$$\begin{aligned}
&\geq \frac{2}{r_0} \left( -L(\lambda, \mu) + \int f_{x_0, \frac{r_0}{2}} d\lambda \right) \\
&\geq \frac{2}{r_0} \left( -r_\lambda + \int_{B(x_0, \frac{r_0}{2})} f_{x_0, \frac{r_0}{2}} d\lambda \right) \\
&\geq \frac{2}{r_0} \left( -r_\lambda + \frac{r_0}{2} \lambda \left( B \left( x_0, \frac{r_0}{2} \right) \right) \right) \\
&= \frac{1}{2} \lambda \left( B \left( x_0, \frac{r_0}{2} \right) \right).
\end{aligned}$$

This shows that  $\mu(K \cap B(x_0, r_0)) > 0$ , and we therefore infer from Lemma 2.3.1 that  $\overline{D}_\mu(\infty) \leq \overline{\dim}_B(K \cap B(x_0, r_0)) < t$ .  $\square$

**Proposition 2.3.3.** *The set  $M^1$  is dense in  $\mathcal{P}(K)$ .*

**Proof.** Let  $\mu \in \mathcal{P}(K)$  and  $0 < \varepsilon < 1$ . Since  $\bar{s} < t$ , there exist  $x_0 \in K$  and  $r_0 > 0$  such that  $\overline{\dim}_B(K \cap B(x_0, r_0)) < t$ . Now put  $\lambda = \frac{\varepsilon}{2} \delta_{x_0} + (1 - \frac{\varepsilon}{2})\mu$ . Since  $\lambda(B(x_0, \frac{r_0}{2})) \geq \frac{\varepsilon}{2} > 0$ , we conclude that  $\lambda \in \Lambda^1 \subseteq M^1$ . Also,

$$L(\mu, \lambda) = \sup_{f \in \text{Lip}(K)} \left| \int f d\mu - \int f d\lambda \right| = \sup_{f \in \text{Lip}(K)} \frac{\varepsilon}{2} \left| \int f d\mu - f(0) \right| \leq \sup_{f \in \text{Lip}(K)} \varepsilon = \varepsilon.$$

This shows that  $M^1$  is dense in  $\mathcal{P}(K)$ .  $\square$

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